Numerical Solution of Overdetermined PDEs

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Stokes Problem

Navier-Stokes are exceptionally useful because they describe the physics of many things of academic and scientific interest. They may be used to model the weather, water flow in a pipe, the air's flow around a wing, motion of stars inside a galaxy.

The Navier-Stokes equations simplify to give the (stationary) Stokes equations:

$$-\mu\Delta u + \nabla p = f$$

$$-\nabla \cdot u = 0$$

where u is the velocity field, p is the pressure and μ is viscosity. By taking divergence of the first equation we obtain

$$-\Delta p = -\nabla \cdot f$$

The new equation is called a differential consequence or integrability condition of the initial system. We call the new system completed system. Putting $y = (u, p)$ we can write the whole system as

$$A_0 y = \begin{cases} -\mu\Delta u + \nabla p = f \\ -\Delta p = -\nabla \cdot f \\ -\nabla \cdot u = 0 \end{cases} \tag{1}$$

This is clearly elliptic. Now A_0 is in general overdetermined, there are no solutions for arbitrary right hand side. There are some compatibility conditions for the solution to exist. These condition are given by an operator A_1 such that $A_1 A_0 = 0$. Such an operator A_1 is called the compatibility operator. The compatibility operator is in the present case given by

$$A_1 = (\nabla\cdot, 1, -\Delta)$$

Let us again introduce a new variable z, and denote $\tilde{y} = (y, z) = (u, p, z)$ and define

$$A\tilde{y} = (A_0, A_1^T)\tilde{y} = A_0 y + A_1^T z$$

we call this system the augmented system. It can be written as

$$A\tilde{y} = \begin{cases} -\mu\Delta u + \nabla p - \nabla z = f \\ -\Delta p + z = -\nabla \cdot f \\ -\nabla \cdot u - \mu\Delta z = 0 \end{cases} \tag{2}$$

We need 4 boundary conditions. For u we use the original boundary condition and for z a natural choice is $z = 0$ on the boundary and Neuman boundary condition for p then we have following boundary condition for augmented system,

$$\begin{cases} \text{u as in the original system} \\ \frac{\partial p}{\partial n} = 0 \\ z = 0 \end{cases}$$

The augmented system is reasonable because it is square and elliptic. This system is explained completely in [1], [2]. There are some numerical examples in these papers. In fact for Stokes system, finite element spaces for velocity and pressure should satisfy in inf-sup condition. On the other hand, augmented system yields a well-posed problem for any finite element spaces.

Implementation

We use Finite Element Method for numerical solution. We have chosen FreeFem++ for our implementation. It is a high level integrated development environment for numerically solving partial differential equations. FreeFem++ is so quick and has a general purpose elliptic solver interfaced with fast algorithms and an advanced automatic mesh generator, capable of a posteriori mesh adaptation. It has several triangular finite elements, including discontinuous elements. Everything is there in FreeFem++ to prepare research quality reports: color display online with zooming and other features and postscript printouts. FreeFem++ is a freeware and this runs on Mac, Unix and Windows.
The main developer of FreeFem++ is Professor Frédéric Hecth in Université Pierre et Marie Curie in Paris. The manual is also available .
The main web page is http://www.freefem.org/ff++.
As an example we consider the Stokes problem in a cavity.

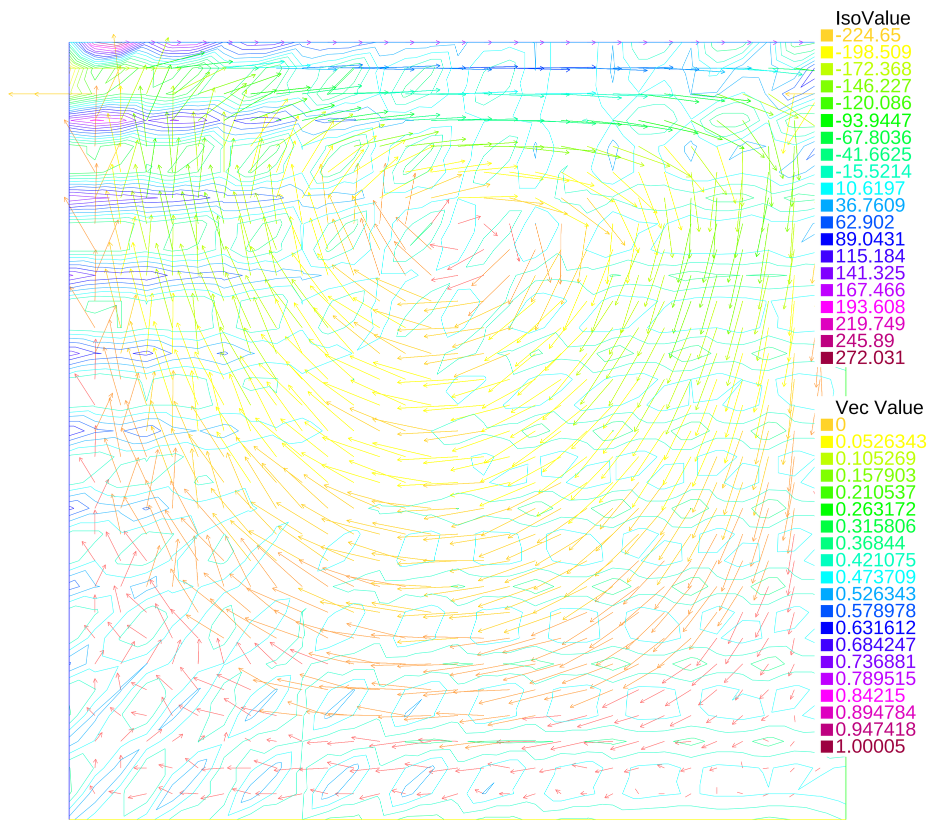

Figure 1: Solution of Stokes equation (1) for the driven cavity problem, showing the velocity field and the pressure level lines. We choose $P2$ for functional space. It is obvious choosing $P2$ for velocity and pressure causes unreasonable result because they do not satisfy in inf-sup condition.

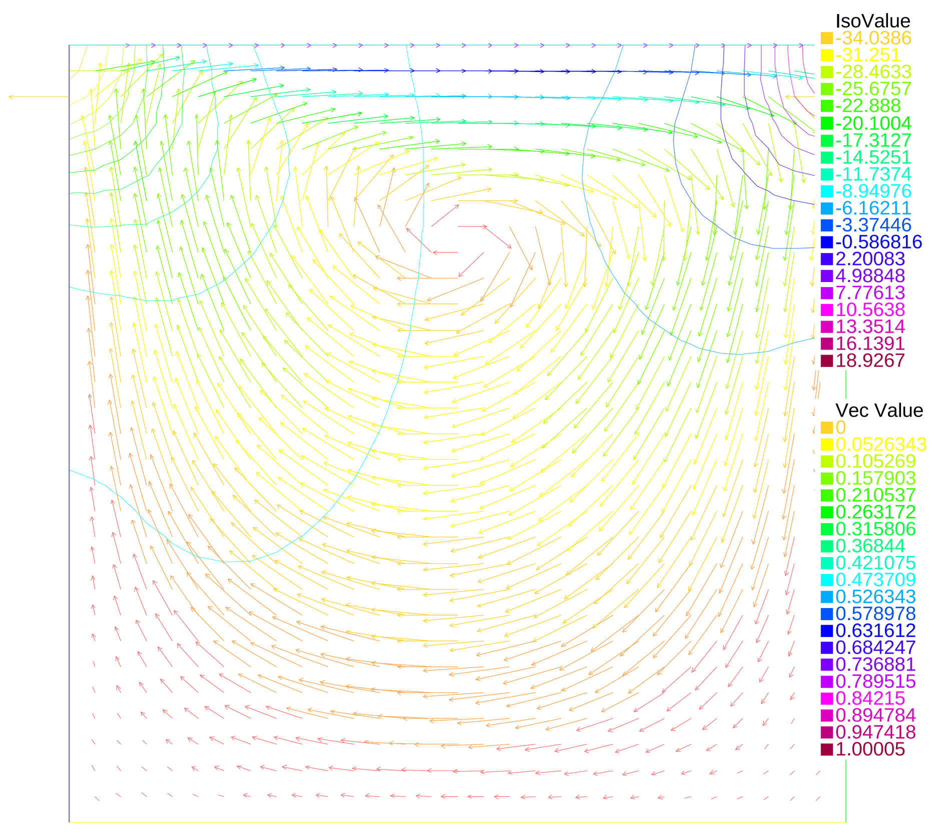

Figure 2: Solution of augmented system (2) for the driven cavity problem, showing the velocity field and the pressure level lines. We can choose $P2$ functional space for velocity and pressure.

We can apply this method also for other overdetermined systems in science and technology.

Future Plan

Microfluid Model :
We deal with a microfluidic flow model where the movement of several charged species is coupled with electric field and the motion of ambient fluid. The main numerical difficulty in this model is the net charge neutrality assumption. It makes the system essentially overdetermined.
Microfluid Model has application in electrophoresis. It is the motion of dispersed particles relative to a fluid under the influence of a spatially uniform electric field.

Koiter shell model:
The mathematical model of the linearly elastic thin shell, in which the unknown is the field of the displacement of the shell middle surface.
There are several variant of Koiter shell model. We will consider the linear case of fourth order, hence quite difficult to discretize. We will reduce the order. It will be overdetermined system. It is not a problem for our method.

References

[1] B. Mohammadi and J. Tuomela. Simplifying numerical solution of constrained PDE systems through involutive completion. *M2AN*, 39(5):909–929, 2005.

[2] B. Mohammadi and J. Tuomela. Involutive upgrades of Navier-Stokes solvers. *Int. J. Comp. Fluid Dyn.*, 23(6):439–447, 2009.