Stochastic Interface Equations

A physicist's short story on "Randomness", Conservation laws and Modeling

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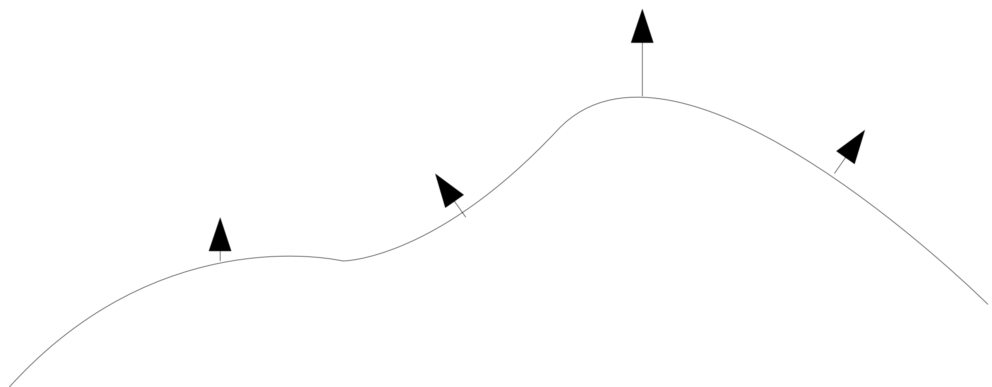
Outline

1) Interface dynamics, experiments and modeling
2) Local Interface Equations (LIEs)
3) Characterizing a stochastic interface, Scaling
4) Case study of Hele-Shaw cells
5) Summary

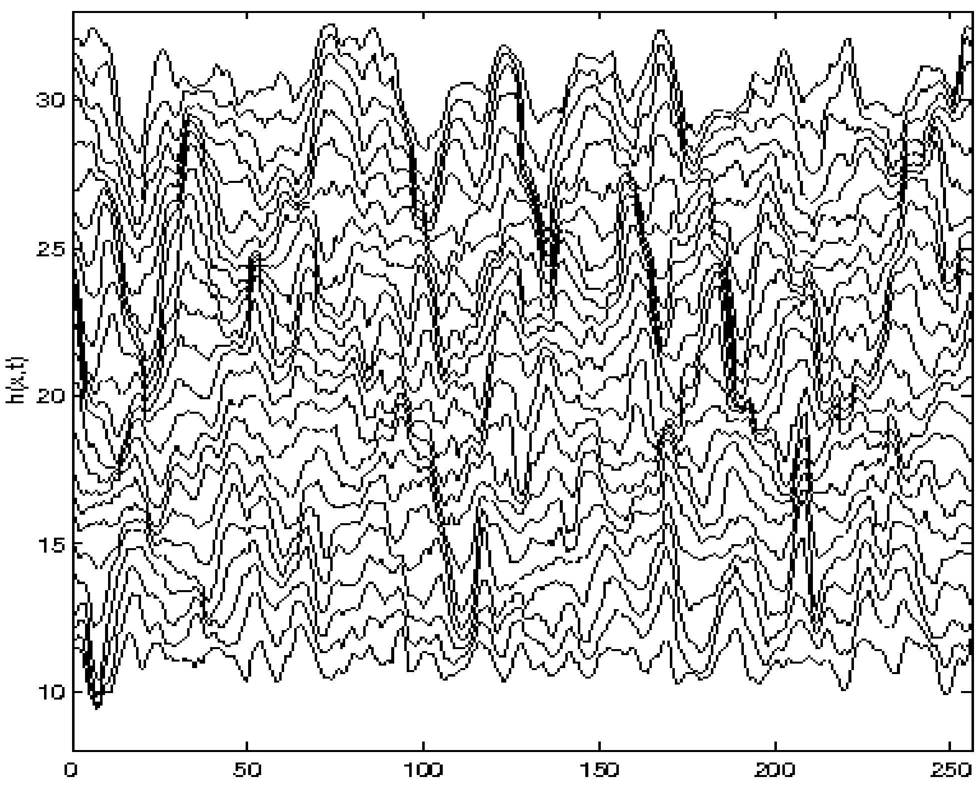
Interface Dynamics



$$h(x,t) \quad ; \quad x \in \mathbb{R}^d$$

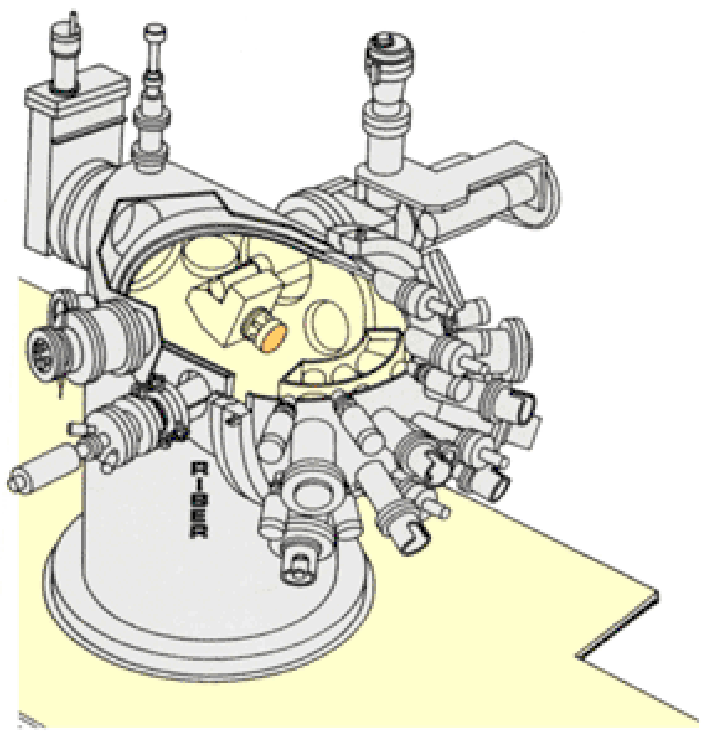
Dimensionality d

Time series:

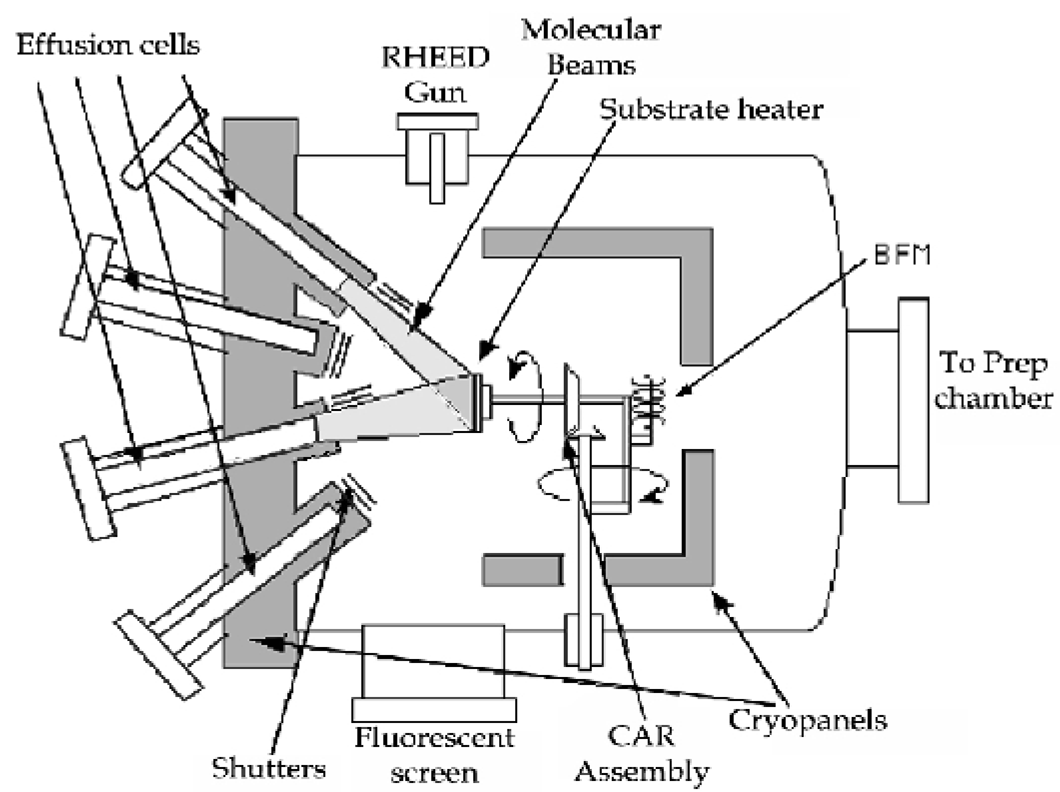


Experiments

- Molecular Beam Epitaxy
  - grow thin films of extreme precision

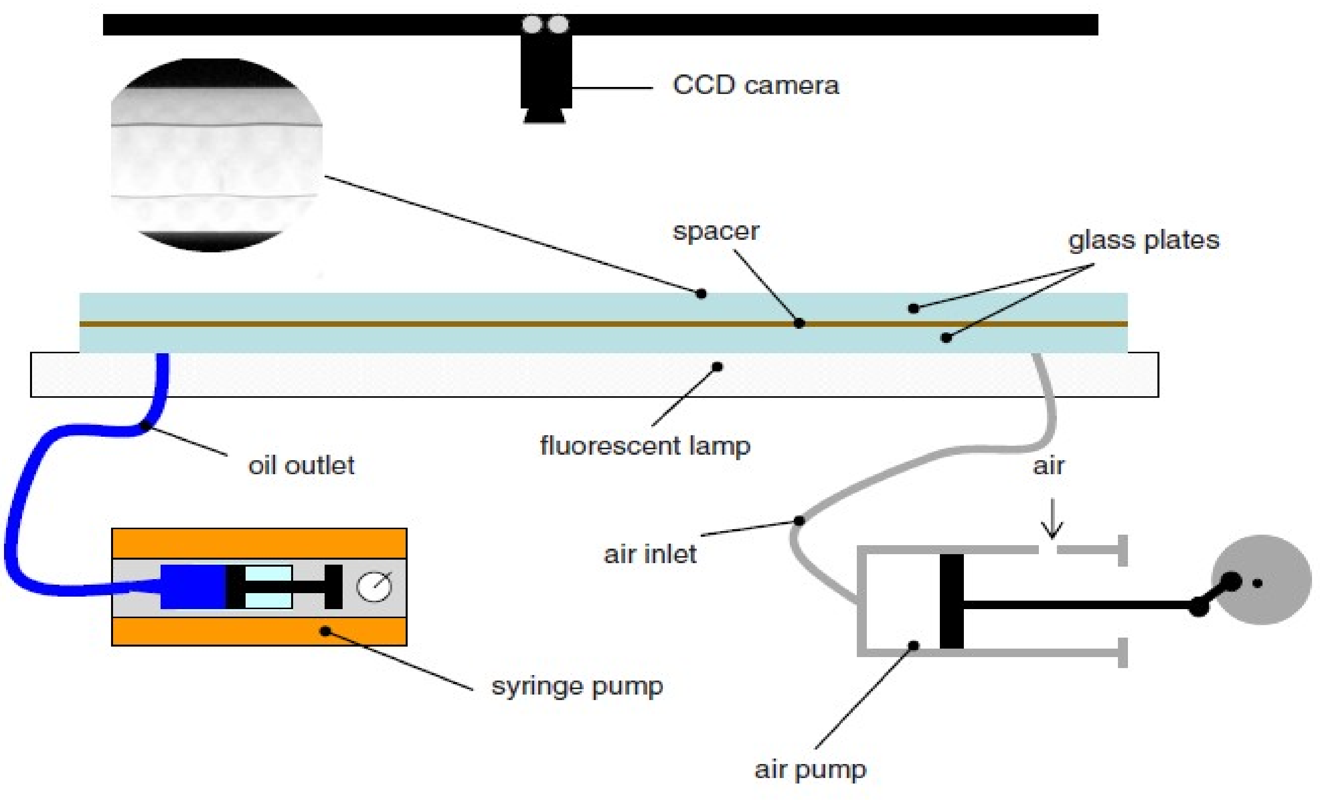


Sketch into a MBE-UHV chamber which shows the (heatable) substrate facing to the effusion cells.



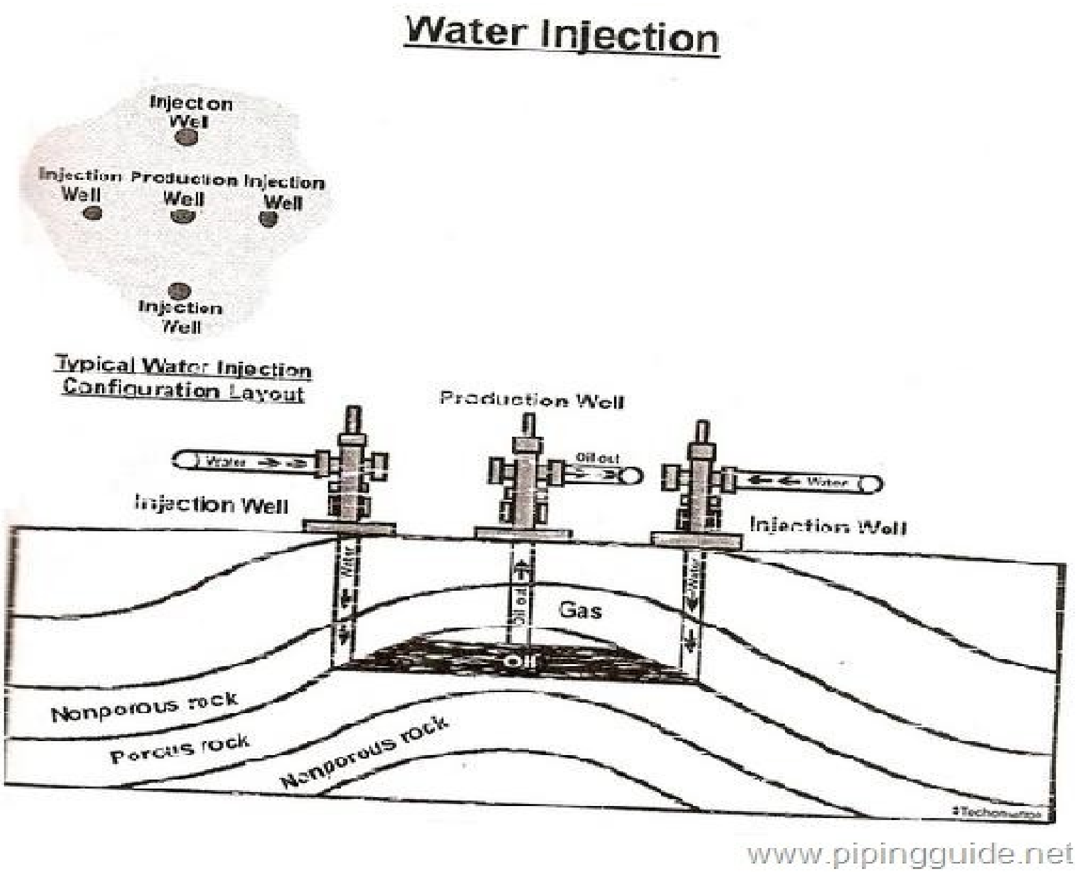
Experiments

- Moving fluid/fluid interfaces (immisible)
  - Laboratory controlled driving and disorder
  - Driven oil/air interface between rough glass plates:

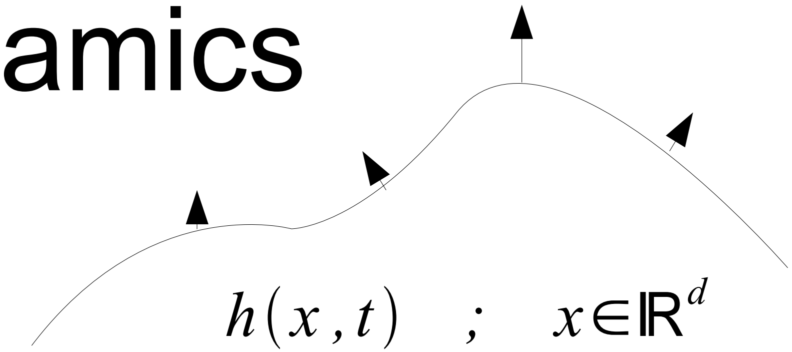


Experiments

- Moving fluid/fluid interfaces (immisible)
  - Industrial scale, rather more uncontrolled
  - Driven oil/water interface :



Interface Dynamics



- Approximations / Modeling
  - Markovian, overdamped

$$\partial_t h(x,t) = D[h(x,t)] + \xi[h(x,t), x, t]$$

D Deterministic part (dispersion, driving)

ξ Stochastic part (disorder/noise)

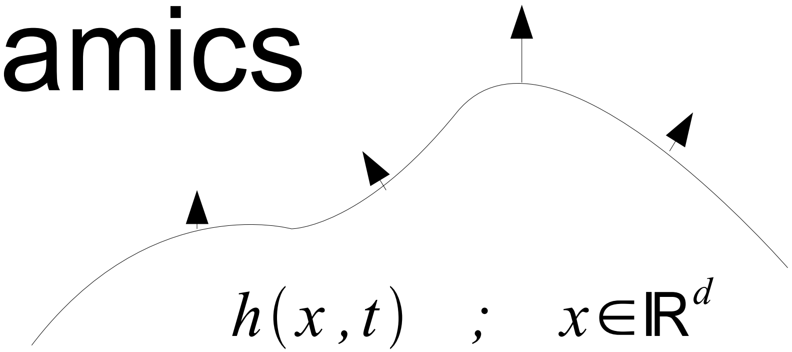
  - Local

$$\partial_t h(x,t) = \sum_{ij} a_{ij} \nabla^i h(x,t)^j + \xi[h(x,t), x, t]$$

  - Linear

$$\partial_t h(x,t) = \sum_i a_i \nabla^i h(x,t) + \xi[h(x,t), x, t]$$

Interface Dynamics



- "Randomness" modeling
  - White noise

$$\xi = \eta(x,t)$$
$$\langle \eta \rangle = 0$$
$$\langle \eta(x,t)\eta(x',t') \rangle = 2\mathrm{D}\,\delta^d(x-x')\,\delta(t-t')$$

  - Correlated noise, e.g.

$$\langle \eta(x,t)\eta(x',t') \rangle = 2\mathrm{D}\,|x-x'|^{2\psi-d}\,|t-t'|^{2\phi-1}$$
$$\langle \eta(k,\omega)\eta(k',\omega') \rangle = 2\mathrm{D}\,k^{-2\psi}\,\omega^{-2\phi}\,\delta^d(k+k')\,\delta(\omega+\omega')$$

  - Disorder field "random landscape"

$$\xi = \eta(x, h(x,t))$$
$$\langle \eta(x,y) \rangle_{x,y} = 0$$
$$\langle \eta(x,y)\eta(x',y') \rangle = 2\mathrm{D}\,\delta^d(x-x')\,\delta(y-y')$$

Local Interface Equations

- Symmetry considerations limit terms
  - Translational invariance (*x,h,t*) only in gradients ∇h

$$\partial_t h(x,t) = \sum_{ij} a_{ij} (\nabla^i h(x,t))^j + \xi[h(x,t), x, t]$$

  - Odd gradients not rotationally symmetric in *x ;* **General**

$$\partial_t h = \nu \nabla^2 h + \lambda/2 (\nabla h)^2 + a_3 \nabla^4 h + a_4 (\nabla^2 h)^2 + a_5 (\nabla^2 h)(\nabla h)^2 + \ldots + \xi$$

  - Up/down symmetry in *h*, no even powers of *h ;* **No driving**

$$\partial_t h = \nu \nabla^2 h + a_3 \nabla^4 h + a_5 (\nabla^2 h)(\nabla h)^2 + \ldots + \xi$$

Local Interface Equations

- Limit of large scales in x means only lowest derivaties that conform to symmetry
- Two most famous local interface equations
  - Edwards-Wilkinson (EW)

$$\partial_t h = \nu \nabla^2 h + \xi$$

  - Kardar-Parisi-Zang (KPZ)

$$\partial_t h = \nu \nabla^2 h + \lambda / 2 (\nabla h)^2 + \xi$$

Hey, what gradients?

- ξ is Wiener process (at best?), h not differentiable

Hey, what gradients?

- ξ is Wiener process (at best), h not differentiable
- Consider discrete form with finite Δx and Δt

$$\langle \eta \rangle = 0$$

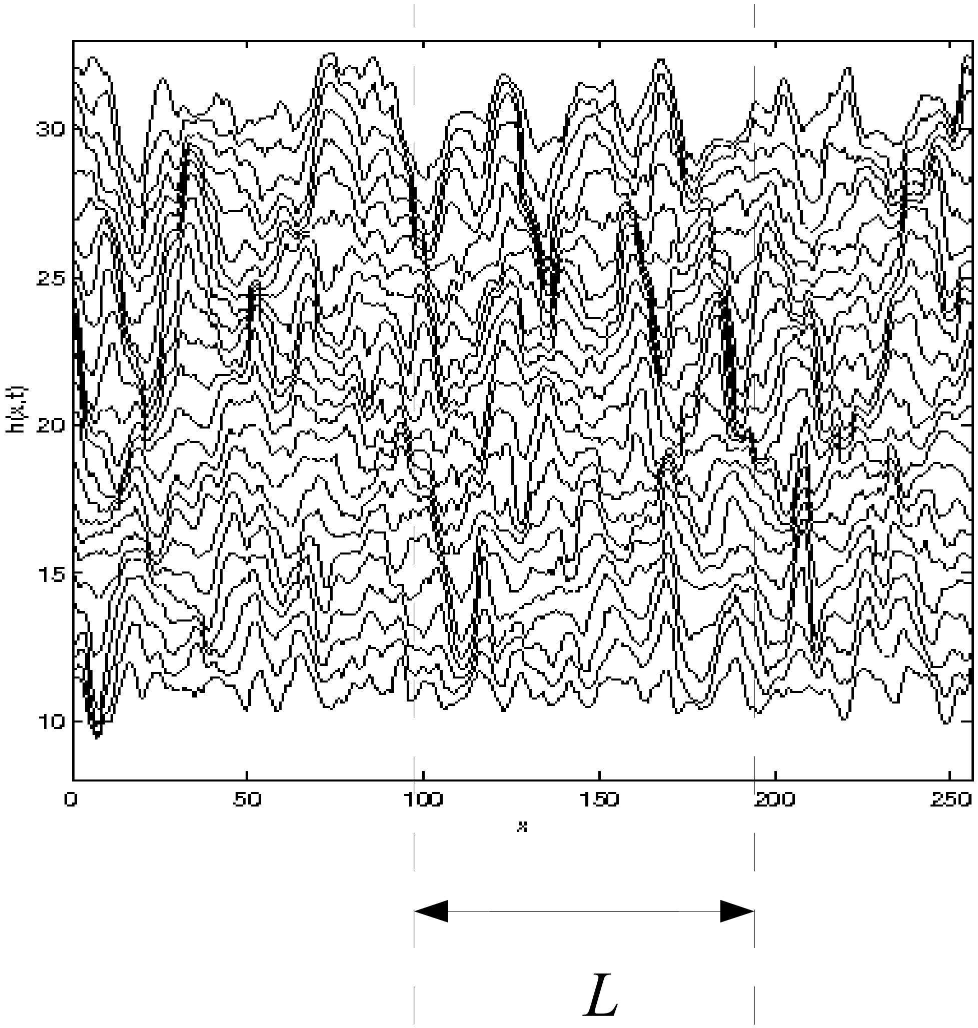
$$\langle \eta(x,t)\eta(x,t')\rangle = \frac{2\mathrm{D}}{\Delta x^d \Delta t}\delta_{xx'}\delta_{tt'}$$

$$\nabla^2 h(x,t) = \frac{1}{\Delta x^2}[h(x-\Delta x) - 2\mathrm{h}(x,t) + h(x+\Delta x,t)]$$

  - Physical matter isn't infinitely continuous
  - Discrete form retains evolution processes (e.g. fluctuation/dissipation, stability/instability)
  - Careful if we want to use mathematical tools - supposedly
  - Δx and Δt can be relevant scales

Characterizing an interface

Time series:



- Width

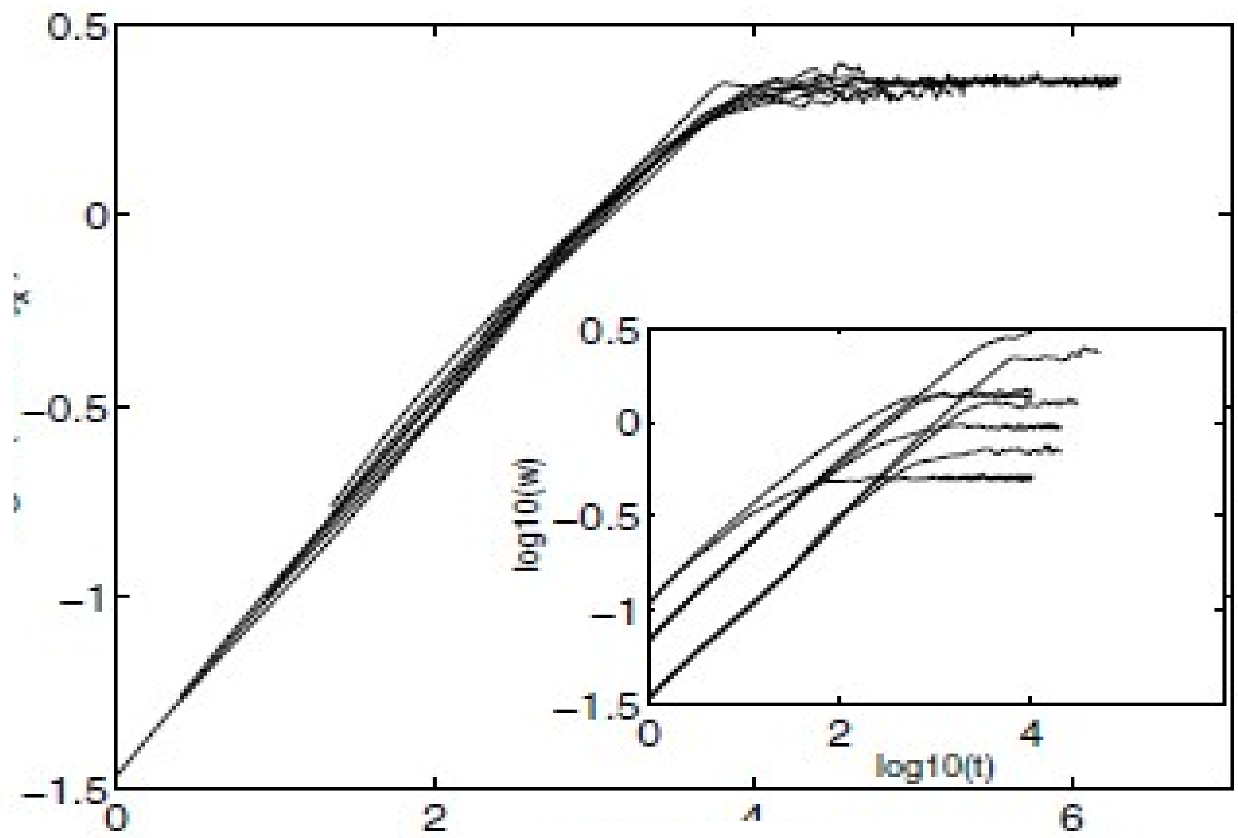
$$w(L,t)=\sqrt{\left\langle \left(h(x,t)-\langle h(t)\rangle_L\right)^2\right\rangle_L}$$

- Scaling (Family/Vicsek)
  - Initially, width grows
    $$w(L,t)\sim t^{\beta}$$
  - Eventually saturates
    $$w_{sat}(L)\sim L^{\alpha}$$
    at time
    $$t_X\sim L^{\alpha/\beta}$$

Scaling

- α Roughness Exponent
- β Growth Exponent

$\log_{10}(w/L^{\alpha})$



$\log_{10}(t/L^{\alpha/\beta})$

EW and KPZ

- Edwards-Wilkinson equation is linear

$$\partial_t h = \nu \nabla^2 h + \xi$$
$$\Leftrightarrow h(k,\omega) = \frac{\xi(k,\omega)}{\nu k^2 - i\omega}$$

  - In case of white noise $\alpha = \frac{2-d}{2} \quad ; \quad \beta = \frac{2-d}{4}$

- Kardar-Parisi-Zhang eq is non-linear

$$\partial_t h = \nu \nabla^2 h + \lambda/2 (\nabla h)^2 + \xi$$

  - No exact solution, even for white noise
  - Dynamic RG is exact in d=1 $\alpha = \frac{1}{2} \quad ; \quad \beta = \frac{1}{3}$
  - Numerical results for d>1

Universality Classes

- Local interface equations (LIE) can be classified to universality classes
  - Assuming large scale limit (lowest gradients only)
  - Assuming white noise
  - Based on conservation of volume under interface
    - Local relaxation conserved/non-conserved
    - Noise conserved/non-conserved
    - Linear/non-linear
  - Linear, conserved $\nabla^2 h$, $\nabla^4 h$
  - Non-linear, conserved $\nabla^2 (\nabla h)^2$
  - Non-linear, non-conserved $(\nabla h)^2$

(Non-)Conserved Noise

$$\langle \eta \rangle = 0$$
$$\langle \eta(x,t)\eta(x',t') \rangle = 2\mathrm{D}\,\delta^d(x-x')\,\delta(t-t')$$

- Non-conserved noise (random additions)

$$\partial_t h(x,t) = D[h(x,t)] + \eta(x,t)$$

- Conserved noise (random potential and flux)

$$\partial_t h = D[h(x,t)] + \nabla\cdot\vec{j} \quad ; \quad \vec{j} = \nabla \eta(x,t)$$
$$\Leftrightarrow \partial_t h = D[h(x,t)] + \nabla^2 \eta(x,t)$$

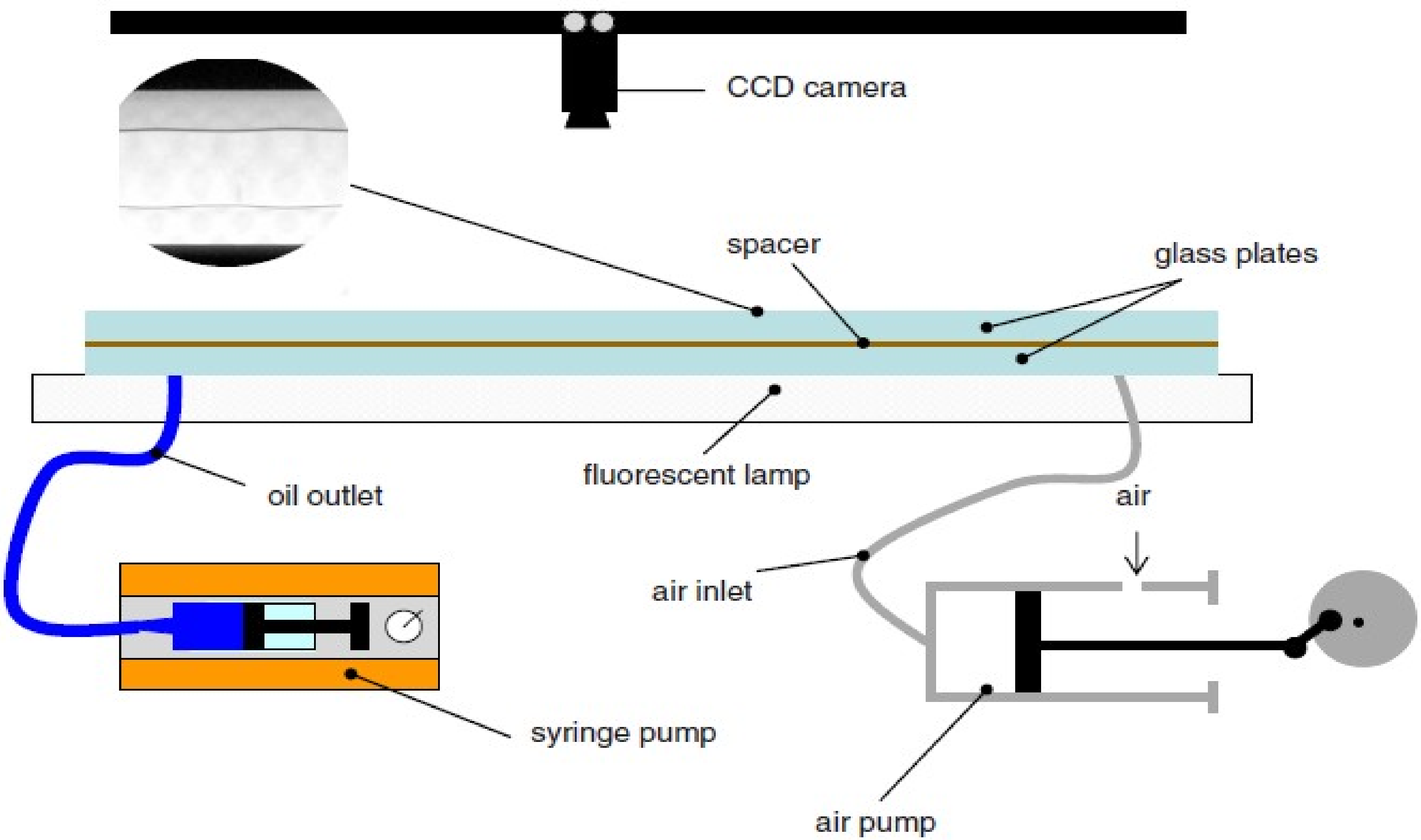
Summary of LIEs

- Assuming
  - Locality of interface relaxation processes
  - Dominance of long wavelength limit
  - Uncorrelated conserved/non-conserved noise
- Interface dynamics can be described by LIEs
- LIEs can be classified to 7 universality classes

- Universality classes describe a wide range of (simple) discrete growth models
- Observed interfaces in nature don't follow these classes "universally"

Case of the Hele-Shaw cell : Liquid/air interface in a rough channel

- Oil is driven to to a narrow, varying gap



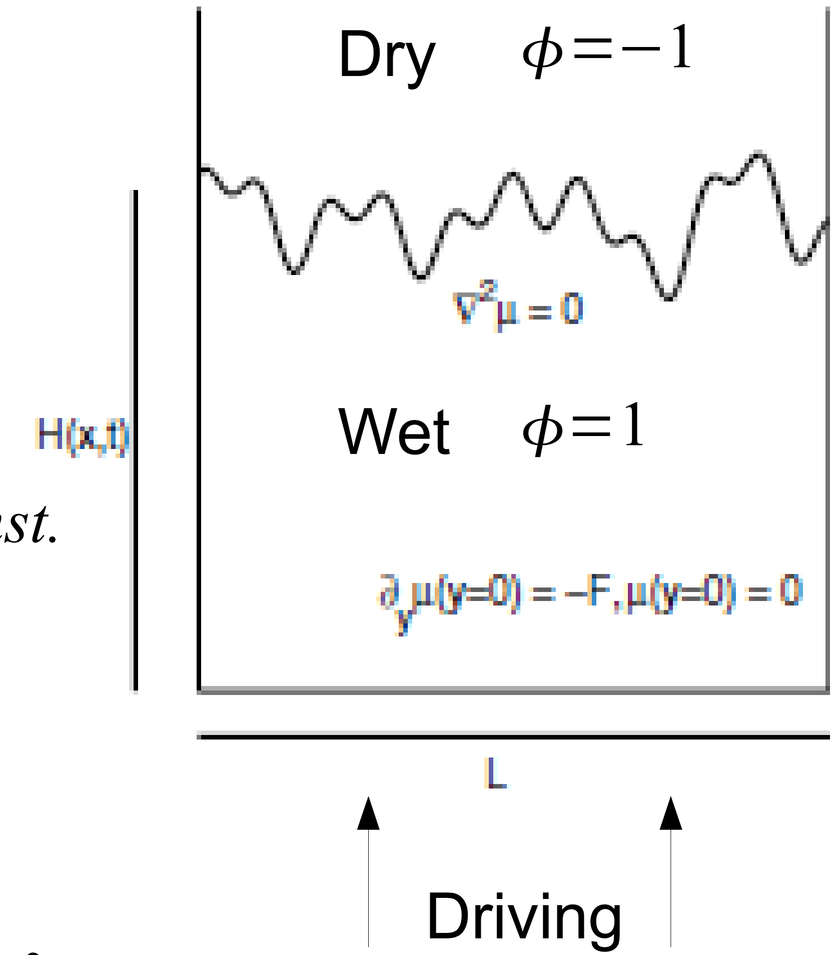
Modeling

- Slow flow: μ ~ pressure
  - Laplace's Eq. in bulk
  - Darcy's law for flow

$$\vec{j} = M \nabla \mu$$

  - Driving at bottom $\nabla \mu = const.$
  - **Mass conservation**

$$\partial_t \phi = \nabla \cdot \vec{j}$$



- Interface between wet/dry

$$\mu = \frac{\delta F}{\delta \phi} + \eta \quad ; \quad F[\phi] = \frac{1}{2} (\nabla \phi)^2 + V(\phi)$$

Random field   Surface tension   Bulk wet or dry

Modeling

- We get a model for slow propagation of oil in channel

$$\partial_t \phi(x,t) = \nabla^2[-\nabla^2 \phi(x,t) + \phi(x,t)^3 - \phi(x,t) + \eta(x)]$$

- $\phi \sim$ oil density ; **Conservation law in 2D**
- Potential flow (=slow), and we can invert ∇^2
- $\phi(x) \sim \tanh(x)$ solves $-\nabla^2 \phi(x,t) + \phi(x,t)^3 - \phi(x,t) = 0$
- Dynamics of small perturbations of interface can be determined with projection techniques

Dynamics of interface fluctuations

- Dispersion relation for small fluctuations becomes

$$H(x,t)=H_0(t)+h(x,t)$$
$$\dot{H}_0=const.$$
$$(1+e^{-2|k|H_0})\partial_t h(k,t)=-[\ |k|(1-e^{-2|k|H_0})\dot{H}_0+\sigma|k^3|\ ]h(k,t)+|k|\eta_k(t)$$

- Or far away from oil injection (Large H)

$$\partial_t h(k,t)=-[\ |k|\dot{H}_0+\sigma|k^3|\ ]h(k,t)+|k|\eta_k(t)$$

Conservation law Capillary waves Effective random field

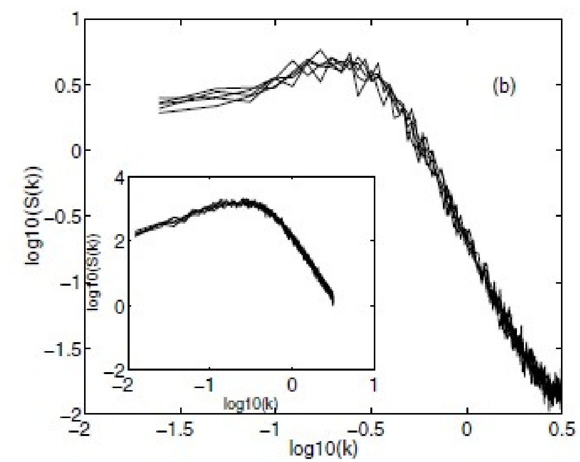
- Mass conservation in 2D ≠ "conserved dynamics"
- Not local in x

Dynamics of interface fluctuations

- Crossover length acts as L

$$\xi_X = 2\pi \sqrt{\frac{\sigma}{\dot{H}_0}}$$

S(k)



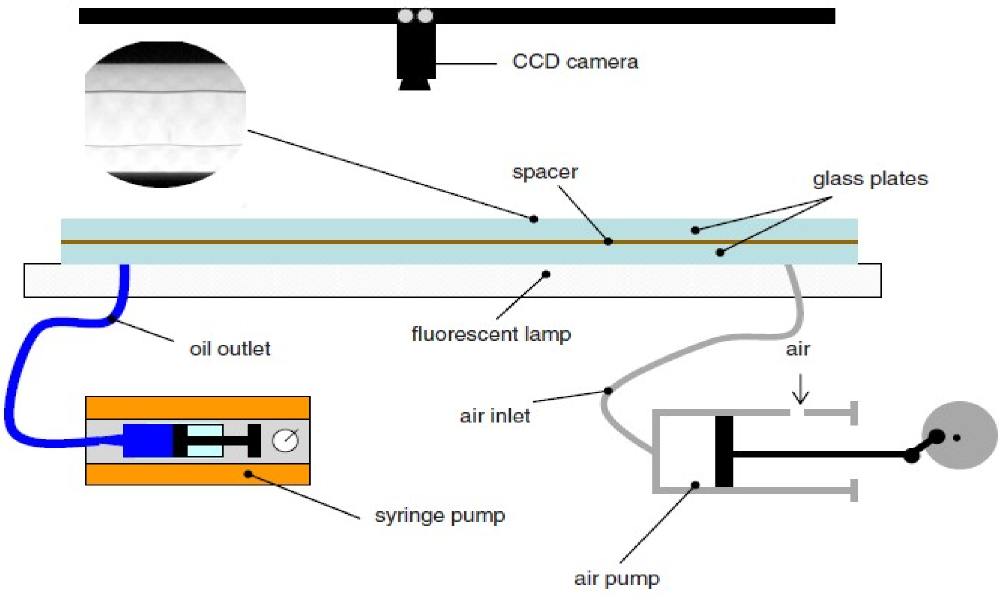
- Interface is super-rough at $dx < \frac{1}{k} < \xi_X$
  Numerically $\alpha \simeq 1.3$
- Effective disorder not local

$$\tilde{\eta}(x,t) = \int dk\, e^{ikx} |k| \eta_k(t) = \int dk\, e^{ikx} |k| \int dx'\, e^{-ikx'} \eta(x', H(x',t))$$

- Local in time because 2D $\Leftrightarrow$ 1D is bijective

More detailed modeling

- Above glass plate roughness modeled as additive stochastic term in potential μ



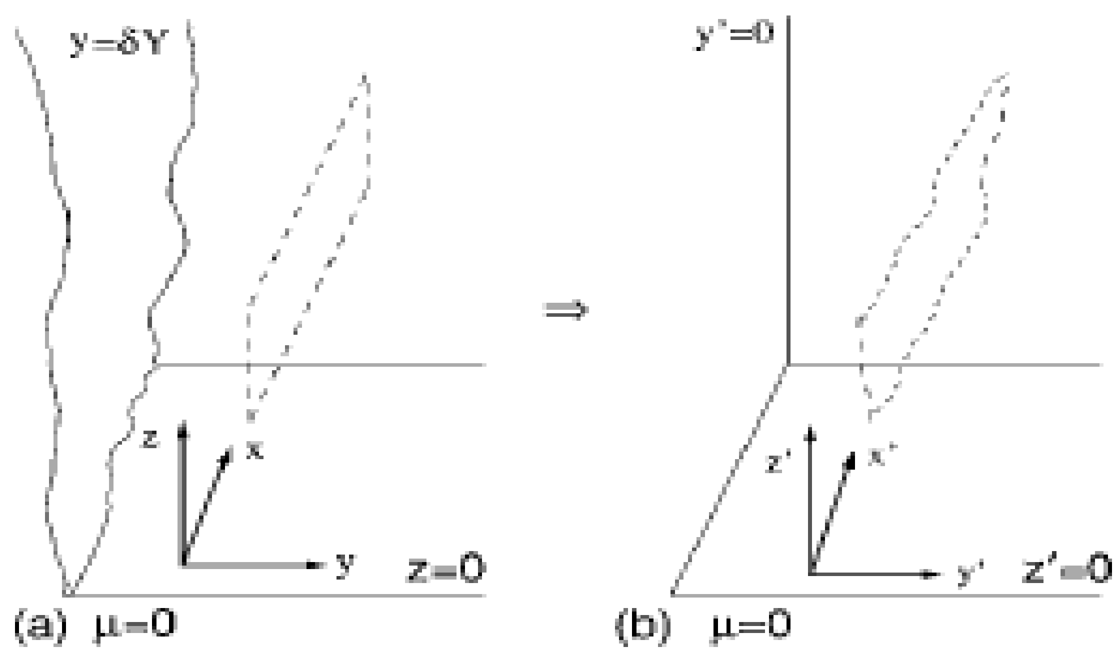
- A stochastic geometry can be modeled directedly

PDE in stochastic geometry

- Our deterministic model in 3D with a rough wall

$$\partial_t \phi(x,t) = \nabla^2[-\nabla^2 \phi(x,t) + \phi(x,t)^3 - \phi(x,t)]$$

- Perturb the Green's function of the Laplacian ∇^2
  - Turns the stochastic boundary location to an algebraic noise term



PDE in stochastic geometry

- Effective noise close to a rough wall in 3D turns out

$$\eta_k(t) = |\vec{k}|\, \dot{H}_0\, \delta\tilde{Y}(\vec{k}, H_0)$$

- Character of modeled *μ* disorder persist
  - 2D model good enough

Summary of oil/air model

- Numerical solution of the two-phase model in agreement with experiments
  - J.Soriano et al. : PRE **67**, 056308 ; PRE **66**, 031603
  - T. Laurila et al. : EPJB **46**, 553 ; PRE **74**, 041601
- Detailed model exposes the difficult character of the interface equation
  - Non-local eqs; Complicated effective stochastic term (even at linear level in perturbations)
- Difficulty result from projection to 1D
  - Describing the interface only in terms of itself embeds processes in bulk to interface dynamics.

Summary

- Morphology and evolution of rough interfaces is characterized by power law scaling
  - Roughness and growth exponents
- Limited number of neatly classified LIEs characterize local interface dynamics
  - Generally real systems are (far) more complex
- Detailed study of physics of Hele-Shaw experiment reveal some insight to the complexity
  - Description of simple dynamics of a complex system turns to complicated dynamics of a simple, coarse system
  - Dynamics of a complex system cause correlated, colored noise when observed at variables of coarse-level model.

More experiments and modeling of disorder

- Recent reviews:
  - Alava, Niskanen, *Physics of Paper,* Reports of Progress in Physics **69**, 669 (2006)
  - Alava, Dube, Rost, Imbibition in Disordered Media, Advances in Physics **113**, 83 (2004).